

On stability of generalised systems of difference equation with non-consistent initial conditions

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Abstract: For given non-consistent initial conditions, we study the stability of a class of generalised linear systems of difference equations with constant coefficients and taking into account that the leading coefficient can be a singular matrix. We focus on the optimal solutions of the system and derive easily testable conditions for stability.

Keywords: singular system, stability, difference equations, optimal, non-consistent.

1 Introduction

Singular systems of difference/differential equation have been studied by many authors in the past years. See [1-28], and [29-36] for recent applications of such systems. For an extended version of this type of systems using fractional operators, see [37-45]. In this article, we consider the following initial value problem:

$$\begin{aligned} FY_{k+1} &= GY_k, \quad k = 1, 2, \dots, \\ Y_0. \end{aligned} \tag{1}$$

Where $F, G \in \mathbb{R}^{m \times m}$ and $Y_k \in \mathbb{R}^m$. The matrix F is singular ($\det F = 0$). The initial conditions Y_0 are considered to be non-consistent. Note that the initial conditions are called consistent if there exists a solution for the system which satisfies the given conditions. We will also assume that the pencil of the system is regular, i.e. for an arbitrary $s \in \mathbb{C}$ we have $\det(sF - G) \neq 0$, see [46-53].

There are stability results in the literature dealing with regular systems and for generalised systems with consistent conditions, see [11, 12, 47, 48]. As already mentioned, we consider initial conditions that are non-consistent. This means that the system has infinite many solutions and an optimal solution is required for this case, see [54, 55]. The aim of this paper is to study the stability of the optimal solution of (1).

2 Preliminaries

Some tools from matrix pencil theory will be used throughout the paper. Since in this article we consider the system (1) with a *regular pencil*, the class of $sF - G$ is characterized by a uniquely defined element, known as the Weierstrass canonical form, see [46-53],

specified by the complete set of invariants of $sF - G$. This is the set of elementary divisors of type $(s - a_j)^{p_j}$, called *finite elementary divisors*, where a_j is a finite eigenvalue of algebraic multiplicity p_j ($1 \leq j \leq \nu$), and the set of elementary divisors of type $\hat{s}^q = \frac{1}{s^q}$, called *infinite elementary divisors*, where q is the algebraic multiplicity of the infinite eigenvalue. $\sum_{j=1}^{\nu} p_j = p$ and $p + q = m$.

From the regularity of $sF - G$, there exist non-singular matrices $P, Q \in \mathbb{R}^{m \times m}$ such that

$$\begin{aligned} PFQ &= \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix}, \\ PGQ &= \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}. \end{aligned} \quad (2)$$

J_p, H_q are appropriate matrices with H_q a nilpotent matrix with index q_* , J_p a Jordan matrix and $p + q = m$. With $0_{q,p}$ we denote the zero matrix of $q \times p$. The matrix Q can be written as

$$Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \quad (3)$$

$Q_p \in \mathbb{R}^{m \times p}$ and $Q_q \in \mathbb{R}^{m \times q}$. The following results have been proved.

Theorem 2.1. (See [1-28]) We consider the systems (1) with a regular pencil. Then, its solution exists and for $k \geq 0$, is given by the formula

$$Y_k = Q_p J_p^k C. \quad (4)$$

Where $C \in \mathbb{R}^p$ is a constant vector. The matrices Q_p, Q_q, J_p, H_q are defined by (2), (3).

Non-consistent initial conditions

The following proposition identifies if the initial conditions are non-consistent:

Proposition 2.1. The initial conditions of system (1) are non-consistent if and only if

$$Y_0 \notin \text{colspan} Q_p.$$

We can state the following Theorem, see [54, 55].

Theorem 2.2. We consider the system (1) with known non-consistent initial conditions. For the case that the pencil $sF - G$ is regular, after a perturbation to the non-consistent initial conditions accordingly

$$\min \|Y_0 - \hat{Y}_0\|_2,$$

or, equivalently,

$$\|Y_0 - Q_p(Q_p^* Q_p)^{-1} Y_0\|_2,$$

an optimal solution of the initial value problem (1) is given by

$$\hat{Y}_k = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* Y_0. \quad (5)$$

The matrices Q_p , J_p are given by (2), (3).

3 Main Results

We will focus on the stability of equilibrium state(s) of homogeneous singular discrete time systems:

Definition 3.1. For any system of the form (1), Y_* is an equilibrium state if it does not change under the initial condition, i.e.: Y_* is an equilibrium state if and only if $Y_0 = Y_*$ implies that $Y_k = Y_*$ for all $k \geq 1$.

The set of equilibrium states for a given singular linear system in the form of (1) are given by the following Proposition, see [11, 12]:

Proposition 3.1. Consider the system (1). Then if 1 is not an eigenvalue of the pencil $sF - G$ then

$$Y_* = 0_{m,1}$$

is the unique equilibrium state of the system (1). If 1 is a finite eigenvalue of $sF - G$, then the set E of the equilibrium points of the system (1) is the vector space defined by

$$E = N_r(F - G) \cap \text{colspan} Q_p.$$

Where N_r is the right null space of the matrix $F - G$, Q_p is a matrix with columns the p linear independent (generalized) eigenvectors of the p finite eigenvalues of the pencil.

Proof. If Y_* is an equilibrium state of system (1), then this implies that for

$$Y_0 = Y_*$$

we have

$$Y_* = Y_k = Y_{k+1}.$$

If 1 is not an eigenvalue of the pencil then $\det(F - G) \neq 0$ and

$$FY_* = GY_*,$$

or, equivalently,

$$(F - G)Y_* = 0_{m,1}.$$

Then the above algebraic system has the unique solution

$$Y_* = 0_{m,1}.$$

which is the unique equilibrium state of the system. If 1 is a finite eigenvalue of the pencil then $\det(F - G) = 0$. If Y_* is an equilibrium state of the system, then this implies that for

$$Y_0 = Y_*$$

we have

$$Y_* = Y_k = Y_{k+1}.$$

This requires that Y_* must be a consistent initial condition which from Proposition 2.1 is equal to

$$Y_* \in \text{colspan} Q_p.$$

Moreover we have

$$FY_* = GY_*,$$

or, equivalently,

$$(F - G)Y_* = 0_{m,1},$$

or, equivalently,

$$Y_* \in N_r(F - G).$$

Hence,

$$Y_* \subseteq N_r(F - G) \cap \text{colspan} Q_p,$$

or, equivalently,

$$E \subseteq N_r(F - G) \cap \text{colspan} Q_p.$$

Let now $Y_* \in N_r(F - G) \cap \text{colspan} Q_p$ then we can consider

$$Y_0 = Y_*$$

as a consistent initial condition and

$$(F - G)Y_* = 0_{m,1}$$

or, equivalently,

$$FY_* = GY_*,$$

where Y_* is solution of the system and combined with $Y_0 = Y_*$ we have $Y_* \in E$, or, equivalently,

$$N_r(F - G) \cap \text{colspan} Q_p \subseteq E$$

The proof is completed.

Theorem 3.1. We consider the system (1) with non-consistent initial conditions. An optimal solution is then given by (5). Then an equilibrium state $Y_* \in E$ is stable in the sense of Lyapounov, if and only if, there exist a constant $c \in (0, +\infty)$, such that $\|J_p^k\| \leq c < +\infty$, for all $k \geq 0$.

Proof. An optimal solution is then given by (5):

$$\hat{Y}_k = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* Y_0.$$

We assume that there exist a constant $c \in (0, +\infty)$ such that $\|J_p^k\| \leq c < +\infty$, for all $k > 0$. Furthermore let an equilibrium state $Y_* \in E$. Then

$$Y_* = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* Y_*$$

and easy we obtain

$$\hat{Y}_k - Y_* = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* (\hat{Y}_k - Y_*),$$

or, equivalently,

$$\hat{Y}_k - Y_* = \begin{bmatrix} Q_p & 0_{m,q} \end{bmatrix} \begin{bmatrix} J_p^k & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} (Q_p^* Q_p)^{-1} Q_p^* \\ 0_{q,m} \end{bmatrix} (\hat{Y}_k - Y_*)$$

If we set $\|Q_p\| = \left\| \begin{bmatrix} Q_p & 0_{m,q} \end{bmatrix} \right\|$ and $\|(Q_p^* Q_p)^{-1} Q_p^*\| = \left\| \begin{bmatrix} (Q_p^* Q_p)^{-1} Q_p^* \\ 0_{q,m} \end{bmatrix} \right\|$. Then by taking norms for every $k \geq 0$ we have

$$\|\hat{Y}_k - Y_*\| \leq \|Q_p\| \|J_p^k\| \|(Q_p^* Q_p)^{-1} Q_p^*\| \|Y_0 - Y_*\|$$

Hence for any $\epsilon > 0$, if we chose $\delta(\epsilon) = \frac{\epsilon}{\|Q_p\| \|(Q_p^* Q_p)^{-1} Q_p^*\| c}$, then

$$\|Y_0 - Y_*\| \leq \delta(\epsilon)$$

implies that for every $\epsilon > 0$

$$\|\hat{Y}_k - Y_*\| \leq \|Q_p\| \|J_p^k\| \|(Q_p^* Q_p)^{-1} Q_p^*\| \|Y_0 - Y_*\|,$$

or, equivalently,

$$\|\hat{Y}_k - Y_*\| \leq \|Q_p\| c \|(Q_p^* Q_p)^{-1} Q_p^*\| \frac{\epsilon}{\|Q_p\| \|(Q_p^* Q_p)^{-1} Q_p^*\| c} \leq \epsilon,$$

or, equivalently,

$$\|\hat{Y}_k - Y_*\| \leq \epsilon.$$

The proof is completed.

Theorem 3.2. The system (1) with non-consistent initial conditions has an optimal solution is then given by (5). Then it is asymptotic stable at large, if and only if, all the finite eigenvalues of $sF - G$ lie within the open disc,

$$|s| < 1.$$

Proof. The system (1) with non-consistent initial conditions has an optimal solution is then given by (5):

$$\hat{Y}_k = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* Y_0.$$

Let a_j be a finite eigenvalue of the pencil with algebraic multiplicity p_j . Then the Jordan matrix J_p^k can be written as

$$J_p^k = \text{blockdiag} \begin{bmatrix} J_{p_1}^k(a_1) & J_{p_2}^k(a_2) & \dots & J_{p_\nu}^k(a_\nu) \end{bmatrix},$$

with $J_p^k \in \mathcal{M}_{p_j}$ be a Jordan block. Every element of this matrix has the specific form

$$k^{p_j} a_j^k.$$

The sequence

$$k^{p_j} |a_j^k|,$$

can be written as

$$k^{p_j} e^{k \ln |a_j|}.$$

The system is asymptotic stable at large, when

$$\lim_{k \rightarrow \infty} \hat{Y}_k = Y_*.$$

Thus this holds if and only if

$$\ln |a_j| < 0,$$

or, equivalently,

$$|a_j| < 1.$$

Then for $k \rightarrow +\infty$:

$$k^{p_j} e^{k \ln |a_j|} \rightarrow 0,$$

or, equivalently,

$$k^{p_j} |a_j|^k \rightarrow 0,$$

or, equivalently, for every $k \geq 0$

$$J_p^k \rightarrow 0_{p,p}.$$

Then for every initial condition Y_0

$$\lim_{k \rightarrow \infty} \hat{Y}_k = 0_{m,1}.$$

The proof is completed.

Conclusions

In this article we focused and provided properties for the stability of the optimal solutions of a linear generalized discrete time system in the form of (1) for given non-consistent initial conditions.

References

- [1] T. M. Apostol; *Explicit formulas for solutions of the second order matrix differential equation $Y'' = AY$* , Amer. Math. Monthly 82 (1975), pp. 159-162.
- [2] Apostolopoulos, N., Ortega, F. and Kalogeropoulos, G., 2015. *Causality of singular linear discrete time systems*. arXiv preprint arXiv:1512.04740.

- [3] R. Ben Taher and M. Rachidi; *Linear matrix differential equations of higher-order and applications*, E. J. of Differential Eq., Vol. 2008 (2008), No. 95, pp. 1-12.
- [4] C. Kontzalis, G. Kalogeropoulos. *A note on the relation between a singular linear discrete time system and a singular linear system of fractional nabla difference equations*. arXiv preprint arXiv:1412.2380 (2014).
- [5] C. Kontzalis, G. Kalogeropoulos. *Homogeneous linear matrix difference equations of higher order: Singular case*. arXiv preprint arXiv:1510.04071 (2015).
- [6] C. Kontzalis, G. Kalogeropoulos. *Solutions of Generalized Linear Matrix Differential Equations with Boundary conditions*. (2015).
- [7] L. Dai, *Impulsive modes and causality in singular systems*, International Journal of Control, Vol 50, number 4 (1989).
- [8] I. Dassios, *On a boundary value problem of a class of generalized linear discrete time systems*, Advances in Difference Equations, Springer, 2011:51 (2011).
- [9] I.K. Dassios, *On non-homogeneous linear generalized linear discrete time systems*, Circuits systems and signal processing, Volume 31, Number 5, 1699-1712 (2012).
- [10] I. Dassios, *On solutions and algebraic duality of generalized linear discrete time systems*, Discrete Mathematics and Applications, Volume 22, No. 5-6, 665-682 (2012).
- [11] I. Dassios, *On stability and state feedback stabilization of singular linear matrix difference equations*, Advances in difference equations, 2012:75 (2012).
- [12] I. Dassios, *On robust stability of autonomous singular linear matrix difference equations*, Applied Mathematics and Computation, Volume 218, Issue 12, 6912-6920 (2012).
- [13] I.K. Dassios, G. Kalogeropoulos, *On a non-homogeneous singular linear discrete time system with a singular matrix pencil*, Circuits systems and signal processing, Volume 32, Issue 4, 1615-1635 (2013).
- [14] I. Dassios, G. Kalogeropoulos, *On the relation between consistent and non consistent initial conditions of singular discrete time systems*, Dynamics of continuous, discrete and impulsive systems Series A: Mathematical Analysis, Volume 20, Number 4a, pp. 447-458 (2013).
- [15] Dassios I., *On a Boundary Value Problem of a Singular Discrete Time System with a Singular Pencil*, Dynamics of continuous. Discrete and Impulsive Systems Series A: Mathematical Analysis, 22(3): 211-231 (2015).
- [16] I. K. Dassios, K. Szajowski, *Bayesian optimal control for a non-autonomous stochastic discrete time system*, Applied Mathematics and Computation, Volume 274, 556-564 (2016).
- [17] E. Grispos, S. Giotopoulos, G. Kalogeropoulos; *On generalised linear discrete-time regular delay systems.*, J. Inst. Math. Comput. Sci., Math. Ser. 13, No.2, 179-187, (2000).

- [18] E. Grispos, *Singular generalised autonomous linear differential systems.*, Bull. Greek Math. Soc. 34, 25-43 (1992).
- [19] Kaczorek, T.; *General response formula for two-dimensional linear systems with variable coefficients.* IEEE Trans. Aurom. Control Ac-31, 278-283, (1986).
- [20] Kaczorek, T.; *Equivalence of singular 2-D linear models.* Bull. Polish Academy Sci., Electr. Electrotechnics, 37, (1989).
- [21] Kalogeropoulos, Grigoris, and Charalambos Kontzalis. *Solutions of Higher Order Homogeneous Linear Matrix Differential Equations: Singular Case.* arXiv preprint arXiv:1501.05667 (2015).
- [22] J. Klamka, J. Wyrwa, *Controllability of second-order infinite-dimensional systems.* Syst. Control Lett. 57, No. 5, 386–391 (2008).
- [23] J. Klamka, *Controllability of dynamical systems*, Matematyka Stosowana, 50, no.9, pp.57-75, (2008).
- [24] C. Kontzalis, G. Kalogeropoulos. *Controllability and reachability of singular linear discrete time systems.* arXiv preprint arXiv:1406.1489 (2014).
- [25] F. L. Lewis; *A survey of linear singular systems*, Circuits Syst. Signal Process. 5, 3-36, (1986).
- [26] F.L. Lewis; *Recent work in singular systems*, Proc. Int. Symp. Singular systems, pp. 20-24, Atlanta, GA, (1987).
- [27] F. Milano; I. Dassios, *Small-Signal Stability Analysis for Non-Index 1 Hessenberg Form Systems of Delay Differential-Algebraic Equations*, Circuits and Systems I: Regular Papers, IEEE Transactions on 63(9):1521-1530 (2016).
- [28] L. Verde-Star; *Operator identities and the solution of linear matrix difference and differential equations*, Studies in Applied Mathematics 91 (1994), pp. 153-177.
- [29] I. Dassios, A. Zimbidis, *The classical Samuelson's model in a multi-country context under a delayed framework with interaction*, Dynamics of continuous, discrete and impulsive systems Series B: Applications & Algorithms, Volume 21, Number 4-5b pp. 261–274 (2014).
- [30] I. Dassios, A. Zimbidis, C. Kontzalis. *The Delay Effect in a Stochastic Multiplier-Accelerator Model.* Journal of Economic Structures 2014, 3:7.
- [31] I. Dassios, G. Kalogeropoulos, *On the stability of equilibrium for a reformulated foreign trade model of three countries.* Journal of Industrial Engineering International, Springer, Volume 10, Issue 3, pp. 1-9 (2014). 10:71 DOI 10.1007/s40092-014-0071-9.
- [32] I. Dassios, M. Devine. *A macroeconomic mathematical model for the national income of a union of countries with interaction and trade.* Journal of Economic Structures 2016, 5:18.
- [33] Ogata, K: Discrete Time Control Systems. Prentice Hall, (1987)
- [34] W.J. Rugh; *Linear system theory*, Prentice Hall International (Uk), London (1996).

- [35] J.T. Sandefur; *Discrete Dynamical Systems*, Academic Press, (1990).
- [36] A. P. Schinnar, *The Leontief dynamic generalized inverse*. The Quarterly Journal of Economics 92.4 pp. 641-652 (1978).
- [37] D. Baleanu, K. Diethelm, E. Scalas, *Fractional Calculus: Models and Numerical Methods*, World Scientific (2012).
- [38] I.K. Dassios, *Optimal solutions for non-consistent singular linear systems of fractional nabla difference equations*, Circuits, Systems and Signal Processing, Springer, Volume 34, Issue 6, pp. 1769-1797 (2015). DOI 10.1007/s00034-014-9930-2
- [39] I.K. Dassios, D. Baleanu, *On a singular system of fractional nabla difference equations with boundary conditions*, Boundary Value Problems, 2013:148 (2013).
- [40] I.K. Dassios, D.I. Baleanu. *Duality of singular linear systems of fractional nabla difference equations*. Applied Mathematical Modeling, Elsevier, Volume 39, Issue 14, pp. 4180-4195 (2015). DOI 10.1016/j.apm.2014.12.039
- [41] I. Dassios, D. Baleanu, G. Kalogeropoulos, *On non-homogeneous singular systems of fractional nabla difference equations*, Applied Mathematics and Computation, Volume 227, 112–131 (2014).
- [42] I. Dassios, *Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations*, Math. Meth. Appl. Sci., 2015, doi: 10.1002/mma.3771.
- [43] I. Dassios, *Stability and robustness of singular systems of fractional nabla difference equations*. Circuits, Systems and Signal Processing (2016). doi:10.1007/s00034-016-0291-x
- [44] T. Kaczorek, *Application of the Drazin inverse to the analysis of descriptor fractional discrete-time linear systems with regular pencils*. Int. J. Appl. Math. Comput. Sci 23.1, 2013: 29–33 (2014).
- [45] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, p. xxiv+340. Academic Press, San Diego, Calif, USA (1999).
- [46] H.-W. Cheng and S. S.-T. Yau; *More explicit formulas for the matrix exponential*, Linear Algebra Appl. 262 (1997), pp. 131-163.
- [47] B.N. Datta; *Numerical Linear Algebra and Applications*, Cole Publishing Company, 1995.
- [48] L. Dai, *Singular Control Systems*, Lecture Notes in Control and information Sciences Edited by M.Thoma and A.Wyner (1988).
- [49] R. F. Gantmacher; *The theory of matrices I, II*, Chelsea, New York, (1959).
- [50] G. I. Kalogeropoulos; *Matrix pencils and linear systems*, Ph.D Thesis, City University, London, (1985).

- [51] Kontzalis, Charalambos P., and Panayiotis Vlamos. *Solutions of Generalized Linear Matrix Differential Equations which Satisfy Boundary Conditions at Two Points*. Applied Mathematical Sciences 9.10 (2015): 493-505.
- [52] I. E. Leonard; *The matrix exponential*, SIAM Review Vol. 38, No. 3 (1996), pp. 507-512.
- [53] G.W. Steward and J.G. Sun; *Matrix Perturbation Theory*, Oxford University Press, (1990).
- [54] Apostolopoulos, N., Ortega, F. and Kalogeropoulos, G., 2016. *The case of a generalised linear discrete time system with infinite many solutions*. arXiv:1610.00927.
- [55] Apostolopoulos, N., Ortega, F. and Kalogeropoulos, G., 2016. *A boundary value problem of a generalised linear discrete time system with no solutions and infinitely many solutions*. arXiv:1610.08277.